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APPLICATION OF A DOMAIN DECOMPOSITION METHOD TO ELASTOPLASTIC PROBLEMS

L. BADEA and P. GILORMINI
Laboratoire de Mécanique et Technologie, ENS de Cachan/CNRS/Université Paris 6, 61 avenue du Président Wilson, 94230 Cachan, France

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Abstract—In this paper, parallel computing is applied to the solution of elastoplastic problems with strain hardening. After a brief discussion of the classical, variational and finite element formulations of the plasticity problem, an algorithm is proposed. It is based on a variant of the Johnson weak formulation and on the Schwarz domain decomposition method. The domain is triangulated and the subdomains are defined as the union of the elements having a node in common. This leads to many, small subproblems, which can be solved in a simple and efficient manner. The algorithm has been implemented on a parallel computer. The numerical results obtained for two examples (a sheared plate and a perforated plate in tension) compare quite favorably with simulations using a classical finite element code. The computing times obtained show that the proposed method utilizes the parallel structure of the computer very efficiently.

1. INTRODUCTION

The aim of this paper is to apply parallel computing to the solution of small deformation elastoplastic problems with strain hardening, using an algorithm based on the Schwarz domain decomposition method. The plasticity problem is posed as a variant of the formulation proposed by Johnson (1978), taking mixed boundary conditions into account (Badea, 1992a). Although Johnson's variational formulation (and the numerical algorithms based on it) is less well-known than the classical ones, it is a suitable framework for the theoretical analysis of the existence and uniqueness of the solution. This formulation also permits the development of numerical algorithms based on the domain decomposition method, which can compete with the classical approaches, especially when parallel computers are used. More detail on the Johnson formulation can be found in Johnson (1976a, b, 1977, 1978), Hlavacek (1980), Hlavacek and Necas (1980), Hlavacek *et al.* (1988), Miyoshi (1985), Panagiotopoulos (1985), and Badea (1992a, b).

In Section 2 the classical formulation is recalled. A brief presentation is then given of the complete and reduced forms of the Johnson formulation. A finite element formulation is derived from the complete form, and the type of elements to be used is discussed. The plasticity problem with zero body forces and zero tractions on the boundary is considered in Section 3. Continuous piecewise linear finite elements are used to approximate the velocity and stress components, and the hardening parameter. The discretized form reduces the problem to finding a saddle point of a convex–concave functional, i.e. the point where it has an infimum for both the stresses and the hardening parameter and a supremum for the velocities. By using the iterative Uzawa method, the solution of the discretized problem is obtained by alternately solving a variational inequality and a variational equation. The Schwarz method is employed to find both the stresses and the hardening parameter from the inequality; the velocities are obtained from the equation. We propose a variant where the domain is divided into subdomains, which are exactly the supports of the test functions of the finite element space. In other words, a subdomain consists of all the triangles which have a common vertex. This introduces many small subproblems which must be solved in each iteration.

The last section is devoted to numerical examples. The proposed subdomain decomposition method has been implemented on an 8-processor parallel computer. Applications are presented for two examples: shearing of a square plate and tension applied to a perforated plate. In both cases, the results of the subdomain method are compared with those given by the ABAQUS finite element code. The examples show that the proposed method makes good use of the parallel structure of the computer.

2. VARIATIONAL AND FINITE ELEMENT FORMULATIONS OF THE PLASTICITY PROBLEM

The *classical elastoplastic problem* is stated as follows for a given domain Ω in $\mathbf{R}^3$ and a time interval $I =]0, T[$ (Ortiz and Popov, 1985, for instance): find the velocities v, the stresses σ and the m-dimensional hardening parameter ξ in $\Omega \times I$ such that:

$$\begin{aligned}
&\mathscr{F}(\sigma, \xi) \leqslant 0 && \text{(yield criterion)}\\
&\varepsilon(v) = \dot{\varepsilon}^{e} + \dot{\varepsilon}^{p} && \text{(strain rate decomposition)}\\
&\dot{\varepsilon}^{e} = A\dot{\sigma} && \text{(Hooke's law)}\\
&\dot{\varepsilon}^{p} = \lambda \frac{\partial \mathscr{F}}{\partial \sigma}(\sigma, \xi) && \text{(normality rule)}\\
&\lambda \mathscr{F}(\sigma, \xi) = 0 && \text{(Kuhn-Tucker condition, with } \lambda \geqslant 0). \qquad (1)
\end{aligned}$$

An additional equation must also be satisfied, which can be derived from the normality rule and depends on the meaning of the hardening parameter ξ. In addition, the equilibrium equations and the boundary conditions must be satisfied by σ and ξ:

$$\begin{aligned}
-\operatorname{div} \sigma &= f \quad \text{in } \Omega \times I,\\
v &= v^{d} \quad \text{on } \Gamma_{v} \times I,\\
\sigma n &= F \quad \text{on } \Gamma_{F} \times I, \qquad (2)
\end{aligned}$$

with zero initial values.

The variational formulation of the problem requires the following functional spaces: $V_2 = [L^2(\Omega)]^3$, $\bar{H} = \{\tau \in [L^2(\Omega)]^9 : \tau = \tau^T\}$, $L = [L^2(\Omega)]^m$, $H = \bar{H} \times L$, $\mathscr{V} = \{v \in [H^1(\Omega)]^3 : v = 0 \text{ on } \Gamma_v\}$, $H_\Gamma = [H^{1/2}(\Gamma)]^3$; and H'_Γ is the dual of H_Γ. We denote by $\langle .,. \rangle$ and $|.|$ the Euclidian scalar product and norm; $\bar{\mathbf{R}}^6$ denotes the set of all the symmetric second-rank tensors; and $l := \varepsilon(v^d)$. The following bilinear form, scalar product and norm are used below:

(i) $a(\tau, \chi) = \int_\Omega < A\tau, \chi > \mathrm{d}\Omega \quad \text{for } (\tau, \chi) \in \bar{H},$

(ii) $(.,.) = \int_\Omega \langle .,. \rangle \, \mathrm{d}\Omega,$

(iii) $\| . \| = \left[\int_\Omega |.|^2 \, \mathrm{d}\Omega \right]^{1/2}.$

Finally, let the two following convex sets also be defined:

(i) $B = \{(\tau, \eta) \in \bar{\mathbf{R}}^6 \times \mathbf{R}^m : \mathscr{F}(\tau, \eta) \leqslant 0\}$
(ii) $P = \{(\tau, \eta) \in H : (\tau, \eta)(x) \in B \text{ a.e. in } \Omega\}$.

Using the above notations, the following variant of the Johnson variational formulation can be stated for the *complete plasticity problem*: find $[(\sigma, \xi), v] : I \to H \times \mathscr{V}$ such that a.e. in I

$$\begin{aligned}
&(\sigma, \xi) \in P\\
&a(\dot{\sigma}, \tau - \sigma) - [\varepsilon(v), \tau - \sigma]_{\bar{H}} + \alpha(\dot{\xi}, \eta - \xi)_L - (l, \tau - \sigma)_{\bar{H}} \geqslant 0 \quad \forall (\tau, \eta) \in P\\
&[\sigma, \varepsilon(w)]_{\bar{H}} = (F, w)_{H'_\Gamma, H_\Gamma} + (f, w)_{V_2} \quad \forall w \in \mathscr{V} \qquad (3)
\end{aligned}$$

with the initial conditions $\sigma(0) = 0$ and $\xi(0) = 0$ a.e. in Ω. This formulation of the problem is obtained from (1) and (2). The constant α is determined by the hardening parameter used. The plasticity problem has a unique solution under some assumptions, which are verified at least for linear isotropic hardening and for kinematic hardening (Badea, 1992a). If the convex set

$$K(t) = \{(\tau,\eta)\in P : [\tau,\varepsilon(w)]_{\bar{H}} = (F,w)_{H'_\Gamma,H_\Gamma} + (f,w)_{V_2} \quad \forall w \in \mathscr{V}\}$$

is considered for each time $t\in I$, the *reduced plasticity problem* may be written: find (σ,ξ): $I \to H$ such that a.e. in I

$$\begin{gathered}(\sigma,\xi)\in K(t),\\ a(\dot{\sigma},\tau-\sigma)+\alpha(\dot{\xi},\eta-\xi)_L \geqslant 0 \quad \forall(\tau,\eta)\in K(t),\end{gathered} \tag{4}$$

with the initial conditions $\sigma(0) = 0$, $\xi(0) = 0$ a.e. in Ω. The meanings of the ξ variable and the α constant, and the equation that must be added to (1), can now be given in the simple case of linear isotropic strain hardening. For a constant hardening rate equal to h, and denoting by p the equivalent plastic strain, we obtain $\xi = p$, $\alpha = h$, and $\lambda = \dot{p}$. The relations are more complex in the non-linear isotropic case (Badea, 1992a) and for kinematic hardening (Johnson, 1978).

Suppose now that the domain Ω has been meshed and that some finite element subspaces $H_h = \bar{H}_h \times L_h$ and V_h of the spaces $H = \bar{H}\times L$ and $\mathscr{V}$, respectively, have been chosen. For practical applications, the above complete and reduced plasticity problems are replaced by finite element forms obtained by substituting the finite element subspaces for the initial spaces. The hardening parameter and the velocities (for the first form) then become unknowns of the problem along with the stresses. By contrast, neither the hardening parameter nor the stresses (in most usual approaches) are regarded as unknowns in the integration schemes developed from the classical formulation of the problem.

The above approaches to the elastoplastic problem have been used in previous studies. Hlavacek (1980) and Hlavacek *et al.* (1988) employed self-equilibrated triangular block-elements for the reduced problem. It should be noted that this formulation cannot consider non-zero velocities at the boundary and cannot yield the velocity field inside the body. Johnson (1977) used triangular elements to solve the complete problem; the stresses and the hardening parameter were approximated by constant values on each triangle and the velocities by continuous piecewise linear functions. Although Johnson (1977) did not include the case of prescribed tractions on the boundary, his approach can be easily extended.

In general, the finite element form of the complete elastoplastic problem has a unique solution (under the same assumptions as the initial problem) if the finite element spaces $\bar{H}_h$ and V_h satisfy the compatibility condition of the mixed finite element approximations (Le Tallec, 1990). The latter states that there must exist $c(h) > 0$ such that,

$$\inf_{w\in V_h}\sup_{\tau\in\bar{H}_h}\frac{|[\tau,\varepsilon(w)]|}{\|\tau\|\;\|\varepsilon(w)\|} \geqslant c(h). \tag{5}$$

If the time interval I is divided into N equal subintervals, the *finite element problem* can be expressed as follows: find $[(\sigma_h^n,\xi_h^n),v_h^n]\in H_h\times V_h$ for $n = 1,\ldots,N$ such that

$$\begin{gathered}\mathscr{F}(\sigma_h^n,\xi_h^n) \leqslant 0,\\ \frac{1}{\Delta t}a(\sigma_h^n-\sigma_h^{n-1},\tau_h-\sigma_h^n)-[\varepsilon(v_h^n),\tau_h-\sigma_h^n]_{\bar{H}_h}+\frac{\alpha}{\Delta t}(\xi_h^n-\xi_h^{n-1},\eta_h-\xi_h^n)_{L_h}(l_h^n,\tau_h-\sigma_h^n)_{\bar{H}_h}\geqslant 0\\ \forall\,(\tau_h,\eta_h)\in H_h : \mathscr{F}(\tau_h,\eta_h)\leqslant 0\\ [\sigma_h^n,\varepsilon(w_h)]_{\bar{H}_h} = (F_h^n,w_h)_{H'_{\Gamma h},H_{\Gamma h}}+(f_h^n,w_h)_{V_{2h}} \quad \forall\,v_h\in V_h,\end{gathered} \tag{6}$$

where $\Delta t = T/N$ is the time step, with $\sigma_h^0 = 0$, $\xi_h^0 = 0$ as initial conditions.

3. AN ALGORITHM FOR PARALLEL COMPUTING

In this paper, the equations of the elastoplastic problem are solved by taking advantage of the parallel structure of modern computers. This is achieved by using a domain decomposition technique, namely the Schwarz method suitably adapted. More precisely, the inequality and the equation of the finite element problem described in the previous section are decoupled using the Uzawa method. Then, they are solved by means of a relaxation process based on the Schwarz method. This original approach to the solution of the problem is another difference between the algorithm proposed here and the iterative methods used in the classical formulation of the problem.

In the following, the body forces f are neglected, and it is assumed that the prescribed surface tractions F, if any, are zero. The velocities, stresses and hardening parameter are approximated by continuous piecewise linear triangular elements. With this choice, the velocities are not uniquely determined because the finite element spaces $\bar{H}_h$ and V_h do not satisfy the compatibility condition (5) for a general triangulation of the domain, although uniqueness is probable when the number of elements is sufficiently large. For these reasons, the finite element problem is stated using the space $\mathcal{V}_h = V_h/\text{Ker}\ (\varepsilon)$ instead of V_h, where $\text{Ker}\ (\varepsilon) = \{w_h \in V_h\colon [\varepsilon(w_h), \tau_h] = 0\ \forall \tau_h \in \bar{H}_h\}$. It has been shown by Badea (1992a) that this choice ensures the compatibility condition. Consequently, we have to solve the following *discretized elastoplastic problem*: find $[(\sigma_h^n, \xi_h^n), v_h^n] \in H_h \times \mathcal{V}_h$ for $n = 1, \ldots, N$ such that

$$\mathscr{F}(\sigma_h^n, \xi_h^n) \leqslant 0$$

$$\frac{1}{\Delta t} a(\sigma_h^n - \sigma_h^{n-1}, \tau_h - \sigma_h^n) - [\varepsilon(v_h^n), \tau_h - \sigma_h^n]_{\bar{H}_h} + \frac{\alpha}{\Delta t}(\xi_h^n - \xi_h^{n-1}, \eta_h - \xi_h^n)_{L_h} - (l_h^n, \tau_h - \sigma_h^n)_{\bar{H}_h}] \geqslant 0$$

$$\forall (\tau_h, \eta_h) \in H_h : \mathscr{F}(\tau_h, \eta_h) \leqslant 0$$

$$[\sigma_h^n, \varepsilon(u_h)]_{\bar{H}_h} = 0 \quad \forall u_h \in \mathcal{V}_h \tag{7}$$

with $\sigma_h^0 = 0$, $\xi_h^0 = 0$ as initial conditions.

For a given n, and dropping subscript h, the Uzawa method applied to the above problem can be expressed as follows (Johnson, 1977): find $[(\sigma_j^n, \xi_j^n), v_j^n]$ for $j \geqslant 1$ such that $\mathscr{F}(\sigma_j^n, \xi_j^n) \leqslant 0$ with

$$\frac{1}{\Delta t} a(\sigma_j^n - \sigma^{n-1}, \tau - \sigma_j^n) - [\varepsilon(v_{j-1}^n), \tau - \sigma_j^n] + \frac{\alpha}{\Delta t}(\xi_j^n - \xi^{n-1}, \eta - \xi_j^n) - (l^n, \tau - \sigma_j^n) \geqslant 0$$

$$\forall (\tau, \eta) \in H_h : \mathscr{F}(\tau, \eta) \leqslant 0, \tag{8}$$

and

$$[\varepsilon(v_j^n), \varepsilon(v)] = [\varepsilon(v_{j-1}^n), \varepsilon(v)] - \rho[\sigma_j^n, \varepsilon(v)] \quad \forall v \in V_h. \tag{9}$$

Here the parameter ρ must be chosen in the $(0, 2\mu/\Delta t)$ interval, μ being the ellipticity constant of the bilinear form a. Thus, we have to solve alternately a variational inequality (8) which leads to the minimization of a quadratic form over a convex set, and a variational equation (9). It is shown in Johnson (1977) that (σ_j^n, ξ_j^n) converges to (σ_h^n, ξ_h^n) when j tends to infinity. The velocities v_j^n will converge to an element of their equivalence class v_h^n. The Schwarz method is used here to find (σ_j^n, ξ_j^n) from (8) and v_j^n from (9). It has been proved in Badea (1989, 1991) that the Schwarz method converges for any number of subdomains for both equations and inequalities. In the proposed algorithm, the domain is divided into subdomains, which are the supports of the test functions, i.e. a subdomain consists of all the triangles which have a common vertex. At each iteration, this requires the solution of a large number of low-dimensional subproblems.

Badea (1991) estimated the error of the Schwarz method using the maximum principle, and two of the conclusions will be emphasized here. First, the error during an iteration is lower on the subdomains nearest to the boundary, and starting from the outer subdomains is the optimal order to run an iteration. This has been verified in the example of the torsion of an elastic–perfectly plastic prismatic bar with a constant section. Second, the method does converge even if an iteration is carried out over all the subdomains simultaneously. The boundary conditions are transferred from one subdomain to another during the next iteration, and the convergence rate is lower. Therefore, the method converges even if some non-disjoint subdomains are considered simultaneously in the computations.

The solving of inequality (8) requires the solution of an algebraic system and a projection, for each node and at each iteration. In the present study, the projection uses the Newton method. For the solution of eqn (9), an algebraic linear system has to be solved for each node and at each iteration. To describe the application of the Schwarz method to (8) and (9), and to simplify the notation, the time increment number n and the iteration number j of the Uzawa method are assumed fixed and will be dropped. Then, supposing that the domain is divided into m subdomains corresponding to the test functions $\phi_1, \ldots, \phi_m$, we have

$$\sigma_j^n = \sigma = \sum_{i=1}^m \phi_i\sigma^i, \quad v_j^n = v = \sum_{i=1}^m \phi_i v^i, \quad \xi_j^n = \xi = \sum_{i=1}^m \phi_i\xi^i, \quad \sigma^{n-1} = s = \sum_{i=1}^m \phi_i s^i,$$

$$\xi^{n-1} = z = \sum_{i=1}^m \phi_i z^i, \quad v_{j-1}^n = u = \sum_{i=1}^m \phi_i u^i, \quad l^n = l = \sum_{i=1}^m \phi_i l^i. \tag{10}$$

Using these notations, (8) and (9) can be written as

$$\frac{1}{\Delta t}[A(\sigma-s),\tau-\sigma] + \frac{\alpha}{\Delta t}(\xi-z,\eta-\xi) - [\varepsilon(u),\tau-\sigma] - (l,\tau-\sigma) \geqslant 0 \quad \forall\,(\tau,\eta)\in P_h \tag{11}$$

and

$$[\varepsilon(v),\varepsilon(w)] = [\varepsilon(u),\varepsilon(w)] - \rho[\sigma,\varepsilon(w)], \quad \forall\, w\in V_h. \tag{12}$$

Therefore, in the iteration $n+1$ of the Schwarz method applied to inequality (11), approximations of σ and ξ are obtained by solving the following m inequalities:

$$\mathscr{F}(\sigma^{n+1i},\xi^{n+1i}) \leqslant 0:$$

$$\begin{aligned}
&\frac{1}{\Delta t}\sum_{j<i}(\phi_j,\phi_i)\langle A(\sigma^{n+1j}-s^j),\tau^i-\sigma^{n+1i}\rangle + \frac{1}{\Delta t}(\phi_i,\phi_i)\langle A(\sigma^{n+1i}-s^i),\tau^i-\sigma^{n+1i}\rangle\\
&+\frac{1}{\Delta t}\sum_{j>i}(\phi_j,\phi_i)\langle A(\sigma^{nj}-s^j),\tau^i-\sigma^{n+1i}\rangle + \frac{\alpha}{\Delta t}\sum_{j<i}(\phi_j,\phi_i)\langle\xi^{n+1j}-z^j,\eta^i-\xi^{n+1i}\rangle\\
&+\frac{\alpha}{\Delta t}(\phi_i,\phi_i)\langle\xi^{n+1i}-z^i,\eta^i-\xi^{n+1i}\rangle + \frac{\alpha}{\Delta t}\sum_{j>i}(\phi_j,\phi_i)\langle\xi^{nj}-z^j,\eta^i-\xi^{n+1i}\rangle\\
&-\sum_j[\varepsilon(\phi_j u^j),\phi_i(\tau^i-\sigma^{n+1i})] - [l,\phi_i(\tau^i-\sigma^{n+1i})] \geqslant 0
\end{aligned}$$

$$\forall\,(\tau^i,\eta^i)\in\bar{\mathbf{R}}^6\times\mathbf{R}^m : \mathscr{F}(\tau^i,\eta^i)\leqslant 0, \tag{13}$$

where i varies from 1 to m. Similarly, the approximations of v in iteration $n+1$ are obtained as the solutions of m equations:

$$\begin{aligned}
&\sum_{j<i}[\varepsilon(v^{n+1j}\phi_j),\varepsilon(\phi_i w^i)] + [\varepsilon(v^{n+1i}\phi_i),\varepsilon(\phi_i w^i)] + \sum_{j>i}[\varepsilon(v^{nj}\phi_j),\varepsilon(\phi_i,w^i)]\\
&\qquad = \sum_j[\varepsilon(u^j\phi_j),\varepsilon(\phi_i w^i)] - \rho\sum_j[\phi_j\sigma^j,\varepsilon(\phi_i w^i)], \quad \forall\, w^i\in\mathbf{R}^3.
\end{aligned} \tag{14}$$

The solution of inequality (13) is obtained in two steps. First, the following linear system of equations is solved with respect to $(\sigma^{n+1/2i}, \xi^{n+1/2i})$:

$$\frac{1}{\Delta t}\sum_{j<i} A_{ij}(\sigma^{n+1j}-s^j)+\frac{1}{\Delta t}A_{ii}(\sigma^{n+1/2i}-s^i)+\frac{1}{\Delta t}\sum_{j>i} A_{ij}(\sigma^{nj}-s^j)-\sum_j C_{ij}u^j-\sum_j l^{ij}=0,$$

$$\sum_{j<i} D_{ij}(\xi^{n+1j}-z^j)+D_{ii}(\xi^{n+1/2i}-z^i)+\sum_{j>i} D_{ij}(\xi^{nj}-z^j)=0, \tag{15}$$

and then $(\sigma^{n+1i}, \xi^{n+1i})$ is obtained by projecting $(\sigma^{n+1/2i}, \xi^{n+1/2i})$ onto the set $B_i=[(\tau^i, \eta^i)\in \bar{\mathbf{R}}^6\times \mathbf{R}^m : \mathscr{F}(\tau^i, \eta^i)\leqslant 0]$ using the norm generated by the matrix A_{ii} defined below. Finally, eqn (14) is equivalent to the following linear system:

$$\sum_{j<i} E_{ij}v^{n+1j}+E_{ii}v^{n+1i}+\sum_{j>i} E_{ij}v^{nj}=\sum_j E_{ij}u^j-\rho\sum_j C_{ji}^T s^j. \tag{16}$$

In the case of plane problems with isotropic hardening, the forms of the matrices and vectors used in (15) and (16) are as follows:

$$A_{ij}=(\phi_i,\phi_j)A=(\phi_i,\phi_j)\begin{bmatrix} \frac{1-\tilde{\nu}}{2G} & 0 & -\frac{\tilde{\nu}}{2G} \\ 0 & \frac{1}{2G} & 0 \\ -\frac{\tilde{\nu}}{2G} & 0 & \frac{1-\tilde{\nu}}{2G} \end{bmatrix},$$

$$C_{ij}=\begin{bmatrix} \left(\phi_i,\frac{\partial\phi_j}{\partial x_1}\right) & 0 \\ \frac{1}{2}\left(\phi_i,\frac{\partial\phi_j}{\partial x_2}\right) & \frac{1}{2}\left(\phi_i,\frac{\partial\phi_j}{\partial x_1}\right) \\ 0 & \left(\phi_i,\frac{\partial\phi_j}{\partial x_2}\right) \end{bmatrix},$$

$$E_{ij}=\begin{bmatrix} \frac{\partial\phi_i}{\partial x_1}\frac{\partial\phi_j}{\partial x_1}(1+\bar{\nu}^2)+\frac{1}{2}\frac{\partial\phi_i}{\partial x_2}\frac{\partial\phi_j}{\partial x_2} & \frac{\partial\phi_i}{\partial x_1}\frac{\partial\phi_j}{\partial x_2}\bar{\nu}^2+\frac{1}{2}\frac{\partial\phi_j}{\partial x_1}\frac{\partial\phi_i}{\partial x_2} \\ \frac{\partial\phi_i}{\partial x_2}\frac{\partial\phi_j}{\partial x_1}\bar{\nu}^2+\frac{1}{2}\frac{\partial\phi_i}{\partial x_1}\frac{\partial\phi_j}{\partial x_2} & \frac{\partial\phi_i}{\partial x_2}\frac{\partial\phi_j}{\partial x_2}(1+\bar{\nu}^2)+\frac{1}{2}\frac{\partial\phi_i}{\partial x_1}\frac{\partial\phi_j}{\partial x_1} \end{bmatrix},$$

$$l^{ij}=(\phi_i,\phi_j)\begin{pmatrix} l_{11}^j \\ 2l_{12}^j \\ l_{22}^j \end{pmatrix},\quad \sigma^j=\begin{pmatrix} \sigma_{11}^j \\ s_{12}^j \\ \sigma_{12}^j \end{pmatrix},\quad v^j=\begin{pmatrix} v_1^j \\ v_2^j \end{pmatrix},\quad D_{ij}=(\phi_i,\phi_j). \tag{17}$$

In the above matrices, G is the shear modulus and $\tilde{\nu}$ (respectively $\bar{\nu}$) is equal to the Poisson ratio ν (respectively 0) in plane strain and $\nu/(1+\nu)$ (respectively $-\nu/(1-\nu)$) in plane stress.

It should be noted that (15) and (16) can be considered as block Gauss–Seidel iterations for the solution of algebraic systems of inequalities and equations, respectively. The dimensions of the matrices in (15) are 3×3 and 1×1 (for the computing of σ and ξ, respectively), and those in (16) are 2×2 (for the computing of v). The matrices A_{ii} and E_{ii} were decomposed as products of upper triangular and lower triangular matrices before the iterative process. As mentioned above, the subdomains where computations are run simultaneously must be disjoint to have a better convergence rate when several processors are used. Consequently, a preprocessor program has been written, which numbers the subdomains (i.e. the nodes) according to the number of processors used.

Thus, supposing a time step n given and starting from $[(\sigma^{n-1}, \xi^{n-1}), v^{n-1}]$, the general scheme of an iteration consists of three loops:

(i) An outer loop for the Uzawa method (8) and (9). In this loop, the iterations are stopped after iteration j (and therefore the program is finished) if $|v_j^n - v_{j-1}^n|$ is small enough.

(ii) Two inner loops, corresponding to the Schwarz method, applied to (8) and (9), respectively:

(a) Inequality (15) and the Newton projection to find an approximation of (σ_j^n, ξ_j^n). At each iteration and at each node, two 3×3 and 1×1 systems must be solved before the projection is performed. The iterations are stopped when the norm of the difference between two consecutive approximations is small enough.

(b) Equation (16) to find an approximation of v_j^n. At each iteration and at each node, a 2×2 system has to be solved. Here again, the iterations are stopped when the solutions of two consecutive approximations are close enough.

4. NUMERICAL EXAMPLES

The algorithm for the parallel solving of elastoplastic problems described in the previous section has been implemented on a parallel computer. It covers the case of two-dimensional problems in either plane strain or plane stress conditions. The latter conditions are considered in the two examples described here, which concern a square sheared plate and a perforated square plate in tension. The behavior of the material is isotropic elastoplastic in both cases; and it uses the von Mises yield condition and linear strain-hardening. The properties of the material are: a Young modulus of 200 GPa, a Poisson ratio of 0.3, an initial yield stress of 200 MPa and a slope of the hardening curve of 2000 MPa. The program was run on an Alliant computer with eight parallel processors, which enables the simultaneous analysis of up to eight subdomains. The same computer was also used to analyse the two examples with the ABAQUS (1989) general-purpose finite element code using three-node and six-node plane stress elements. The values obtained and the computing time required were compared.

In Section 3, it was emphasized that the rate of convergence of the proposed algorithm depends on the order in which the subdomains are considered during an iteration of the Schwarz method. Therefore, the subdomain numbering of a given structure must be kept unchanged when variations of computing time with respect to the number of processors used are studied. The numbering of the subdomains (i.e. nodes) was chosen to define groups of eight disjoint successive subdomains. Consequently, the subdomains analysed simultaneously during an iteration do not overlap if the number of processors is one, two, four or eight. The efficiency of the parallelism is discussed and diagrams show the computing time t versus $1/n$, where n is the number of processors used. The results for $n = 1, 2, 4$ and 8 were fitted with the least squares method to give

$$t = a + b\frac{1}{n} \tag{18}$$

where a is the non-parallelizable computing time, and b is the part that can be parallelized. In other words, the computing time with a classical sequential computer would be $a+b$, and it would reduce to a with an ideal parallel computer with an infinite number of processors. Consequently, the parallelizable fraction of the computing time is $p = b/(a+b)$.

4.1. *The sheared plate*

A square plate of unit side is considered. The lower edge is fixed and a constant tangential velocity of 0.01 is applied to the upper side. The two other edges are free boundaries. At the end of the time interval, therefore, the upper side is translated by 0.01 along its own direction. Four regular meshes with 98, 450, 1058, and 1922 identical triangles

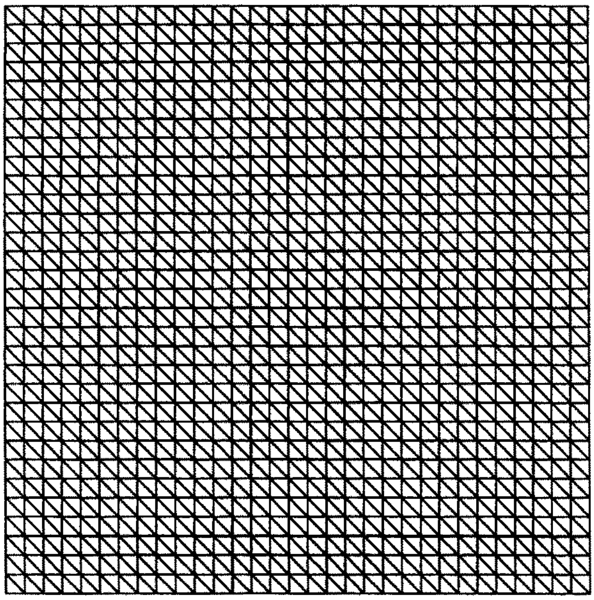

Fig. 1. The 1024-node mesh used in the sheared plate example.

(i.e. 64, 256, 576 and 1024 nodes for three-node elements) were used to study the influence of the problem size upon computing time and results. For instance, Fig. 1 shows the 1922-element case. In the domain decomposition algorithm proposed here, there are six unknowns for each node (two velocity components, three stress components and one hardening parameter). Taking the boundary conditions into account, the total numbers of unknowns in the four problems considered are slightly less than six times the number of nodes; and they equal 342, 1472, 3360 and 6016, respectively, since only three-node elements were used with this method.

An example of computing time obtained, depending on the number of processors used, is shown in Fig. 2. It can be observed that it obeys the linear rule of eqn (18). For the domain decomposition method, the results for three, five, six or seven processors are not far from the line fitted to the results for one, two, four and eight processors. This means that simultaneously analysing non-disjoint subdomains at some steps of the calculation

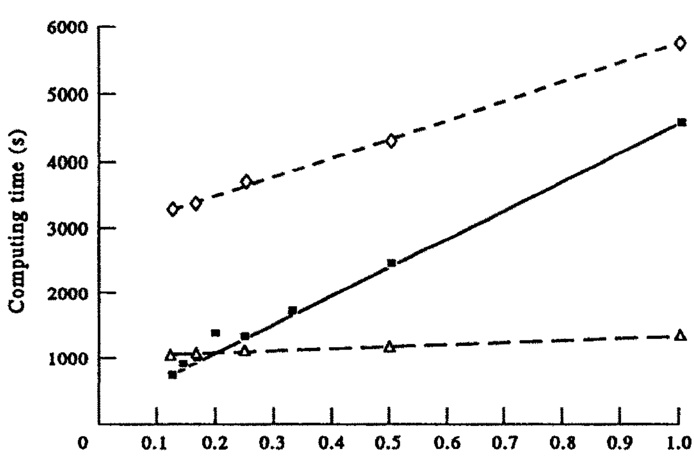

Fig. 2. Computing time versus the inverse of the number of processors used, for the mesh in Fig. 1. Solid symbols: domain decomposition method, open symbols: ABAQUS results (triangles: three-node elements, squares: six-node elements).

does not significantly alter the convergence rate. The slope of the lines in Fig. 2 is directly related to the advantage the algorithm takes from the parallel architecture of the computer. It is clearly seen that the larger the number of variables (six-node versus three-node elements), the more ABAQUS takes advantage of parallelism. However, it is also clear that the domain decomposition method is more efficient from this point of view.

Similar results were obtained for the three other regular meshes. Table 1 gives the values obtained for parameters a and b of the linear rule (18) in the four cases, and the corresponding p value. It is probable that the domain decomposition method would be parallelized up to 90% with a very large number of processors (but this prediction could not be checked definitely with our eight-processor Alliant computer). This figure is independent of the problem size. By contrast, the algorithm used by ABAQUS has a low p value, which increases with the number of degrees of freedom. It should be noted that the foregoing discussion is about *variations* of computing time when different numbers of processors are used, and that *values* of computing time obtained with the two approaches cannot be reasonably compared. This is due to (i) the different time steps used (eight equal time steps in the domain decomposition method, an automatically defined variable time step in ABAQUS), (ii) differences in the number and nature of the unknowns for a given mesh, and (iii) the tests used to stop the calculations, which differ in the two cases.

Figure 3(a) displays the deformed geometry given by the domain decomposition method (with magnified displacements), which helps clarify the boundary conditions prescribed. Figure 3(b) shows the variations of the von Mises equivalent stress at the end of the loading process, and should be compared with Figs 3(c) and 3(d) obtained with the ABAQUS code. Note that two areas centered at the middle of the free edges remain elastic, and that stress singularities occur at the four corners. Consequently, the maximal values obtained increase when the mesh is refined. The results given by the domain decomposition method compare very favorably with those given by the finite element code. A more detailed analysis of the results gives a slightly better agreement when six-node elements are used: the maximal values are closer. Note that ABAQUS computes the stresses at the center of the triangles and that nodal interpolated values are used in Figs 3(c) and 3(d). This leads to smoother contour lines than with the domain decomposition method, where nodal values are obtained directly.

4.2. *The perforated plate in tension*

A square plate is considered, with a side length of 2.0 and with a central hole of radius 0.2. A uniform and normal displacement of 0.01 is applied to two opposite edges with a constant velocity during the time interval]0, 1[(16 equal time steps are used in the domain decomposition method). The other two edges, as well as the central hole, are free boundaries, and by symmetry it suffices to model one-quarter of the plate only (Fig. 4). Contrary to the mesh used in the previous example, the triangulation is not regular, as shown in Fig. 4. This mesh has 917 triangular elements and 508 nodes, resulting in about 3000 unknowns for the domain decomposition method. It was generated automatically, using a procedure which guarantees that the error in the finite element solution for an elastic analysis would be lower than a prescribed value (2% in the present case). More detail on this technique, including the definition of the error calculated, can be found in Coorevits *et al.* (1992).

Figure 5 shows the amount of computing time used for the various number of processors. It can be observed that the points corresponding to three, five, six and seven processors are closer to the straight line (with $a = 82.1$ and $b = 2806.9$ seconds) fitted to

Table 1. Coefficients a and b (seconds) of the law for describing the computing time, and parallelizable fraction p (%)

Number of nodes	a	b	p
64	6.2	83.2	93
256	73.6	997.4	93
576	150.6	2470.6	94
1024	229.4	4377.5	96

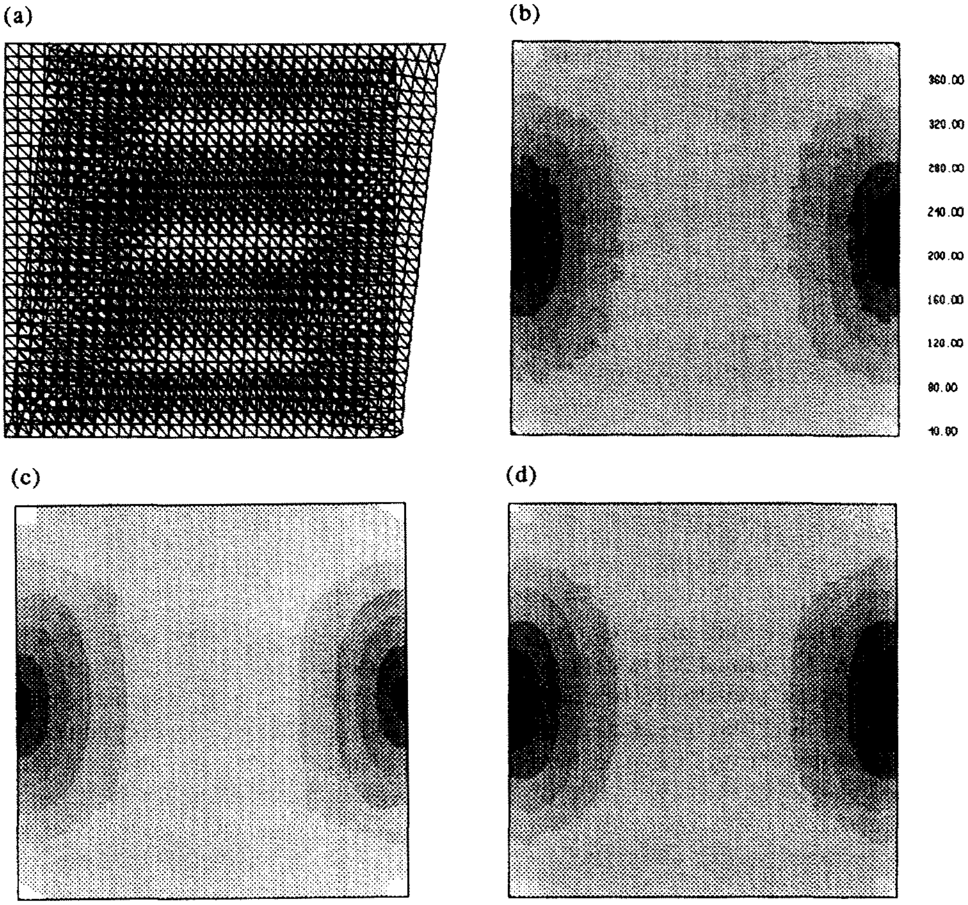

Fig. 3. Results obtained with the mesh shown in Fig. 1. Deformed geometry (a) (the displacements are multiplied by 20) and map of the von Mises equivalent stress given by the domain decomposition method (b). Comparison with the von Mises values given by the ABAQUS code with three-node (c) and six-node (d) triangles.

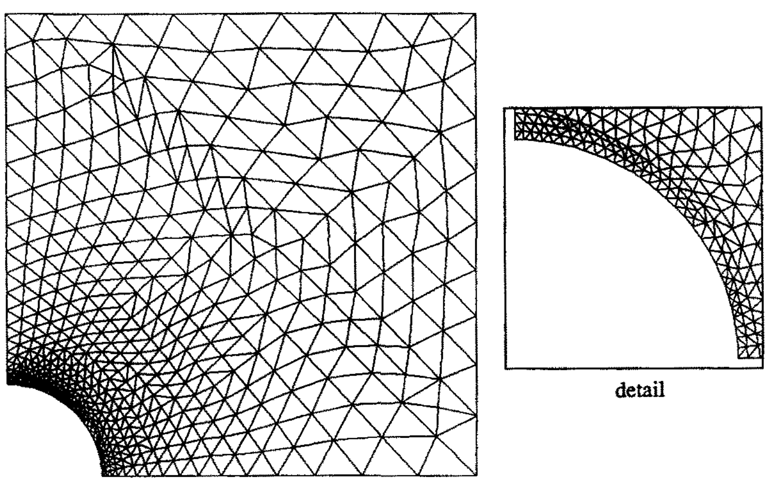

Fig. 4. Mesh used for the perforated plate in tension, with 917 elements and 508 nodes. The box shows a closer view of the hole area.

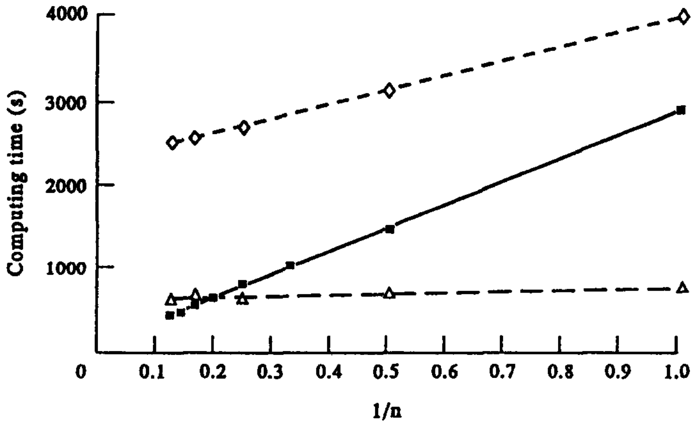

Fig. 5. Computing time versus the inverse of the number of processors used, for the mesh in Fig. 4. Solid symbols: domain decomposition method, open symbols: ABAQUS results (triangles: three-node elements, squares: six-node elements).

the cases of one, two, four or eight processors than in Fig. 2. This may be due to a larger number of time steps in the present case. Comparison with computing times required by ABAQUS using the same mesh with three-node and six-node triangles leads to the same conclusions as in the previous example: the domain decomposition method makes better use of the parallel computer. The fraction p of computing time that can be parallelized is again very high in the subdomain method, and reaches 97%. Due to the program that has been written to number the subdomains (or, equivalently, the nodes), there is no difference between a regular and an irregular mesh in terms of overlapping subdomains simultaneously analysed. This explains why the performances of the domain decomposition method are similar in the two examples presented, although the meshes appear very different.

In order to give some detail on the convergence rate of our algorithm, it may be said that the initial trial values were taken equal to zero in the first time step, where the solution corresponding to the elastic limit was computed. The initialization at the beginning of any further time step used the values obtained at the end of the previous one. In all loops, the absolute errors on the last two approximations were computed. Convergence was obtained when the errors on the velocities obtained in the outer loop were less than 5×10^{-5}. The absolute precisions for stresses and for the hardening parameter in the inner loop corresponding to the inequality were 2×10^{-2} and 2×10^{-6}, respectively. In the inner loop corresponding to the equation, the precision on the velocities was 2×10^{-6}. Taking into account the order of magnitude of the various unknowns (velocities, stresses and hardening parameter), the relative error was about 2×10^{-5} in the inner loops and 5×10^{-4} in the outer loops. For the 16 time steps considered, the mean number of outer iterations was almost constant at about 16.5 per time step. For an outer iteration, the mean number of inner iterations was 48.7 (almost constant for any time step) in the loops corresponding to the equation, and only three for those corresponding to the inequality (this number was about five times greater in the first three time steps than in the following ones, where the initialization procedure is more efficient). Similar conclusions apply to the sheared plate example.

The aspect of the deformed plate (with amplified displacements) is shown in Fig. 6(a). The maps of the von Mises equivalent stress obtained with the domain decomposition method and with ABAQUS using linear and quadratic triangular elements are presented in Figs 6(b), 6(c) and 6(d), respectively. Note that there remains an elastic area along the lower side in Fig. 6, although the plastic region includes the upper-right corner. There is very good agreement between the predictions of the subdomain method and those given by ABAQUS, especially when six-node triangles are used. For instance, the peak von Mises value (obtained near the upper part of the hole) is 373.3 MPa with the subdomain method, while ABAQUS gives 363.9 and 370.5 MPa with three-node and six-node elements, respectively. Finally, Fig. 7 presents the history of the von Mises equivalent stress given by the

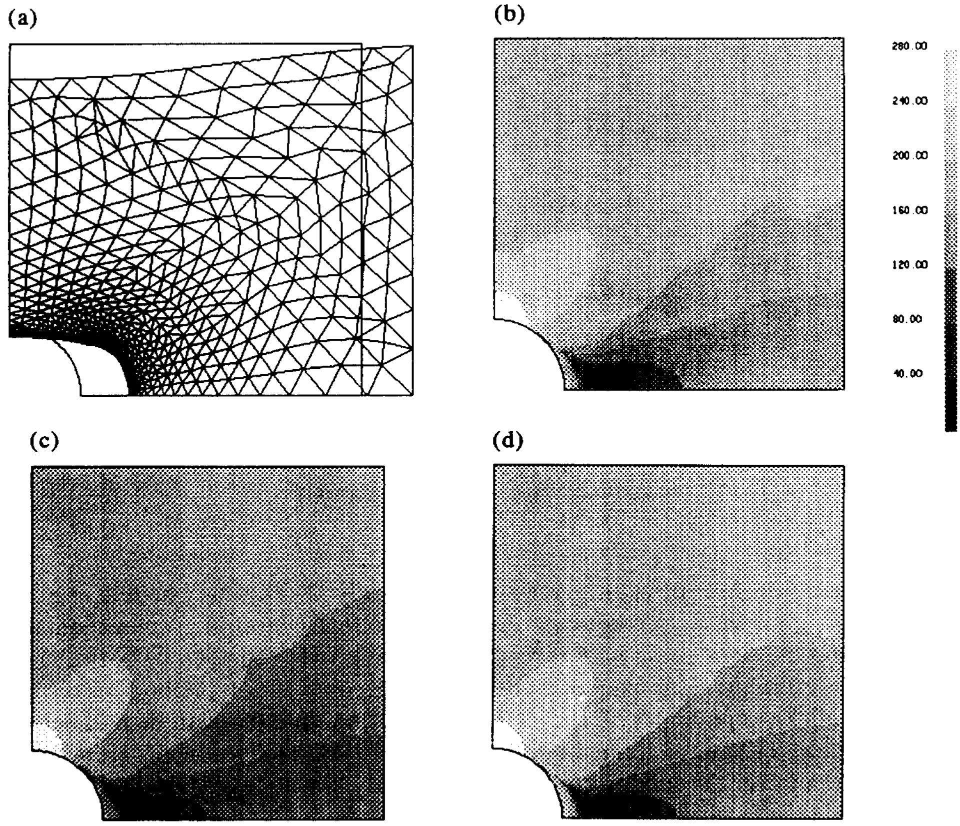

Fig. 6. Results obtained with the mesh shown in Fig. 4. Deformed geometry (a) (the displacements are multiplied by 20) and map of the von Mises equivalent stress given by the domain decomposition method (b). Comparison with the von Mises values given by the ABAQUS code with three-node (c) and six-node (d) triangles.

subdomain decomposition method. In Fig. 7(a), the applied displacement is small and the stress state is elastic in the whole plate (the peak value is 200.0 MPa, i.e. the elastic limit). In Figs 7(b)–7(d), plastic yielding has occurred and the plastic zone covers about two-thirds of the plate, with increasing peak values: 284.5, 339.8, and 373.3 MPa in Figs 7(b), 7(c), and 7(d), respectively.

5. CONCLUSIONS

(i) In this paper, a domain decomposition method has been applied to elastoplastic problems with strain hardening. The finite element form is deduced from the weak formulation of the problem. For each time increment, the Usawa method is used. It requires solutions for a variational equation and a variational inequality, for which a variant of the Schwarz method is employed. The latter method uses the supports of the test functions of the finite element space. Therefore, the subdomains are defined by all the elements having a common node. Because of this definition of the subdomains, the Schwarz method appears as a relaxation method.

(ii) A program has been written to apply this incremental and iterative algorithm. It has been tested in two cases: a square plate in shear (with four meshes of different fineness) and a perforated plane in tension. In both cases, the results obtained are in very good agreement with the predictions of the ABAQUS code.

(iii) Even when only one processor is used, the computing time required by the proposed method is reasonable, although it uses a very large number of subdomains. This is due to the small size of the problem to be solved on each subdomain. Consequently, very simple and efficient numerical methods can be used.

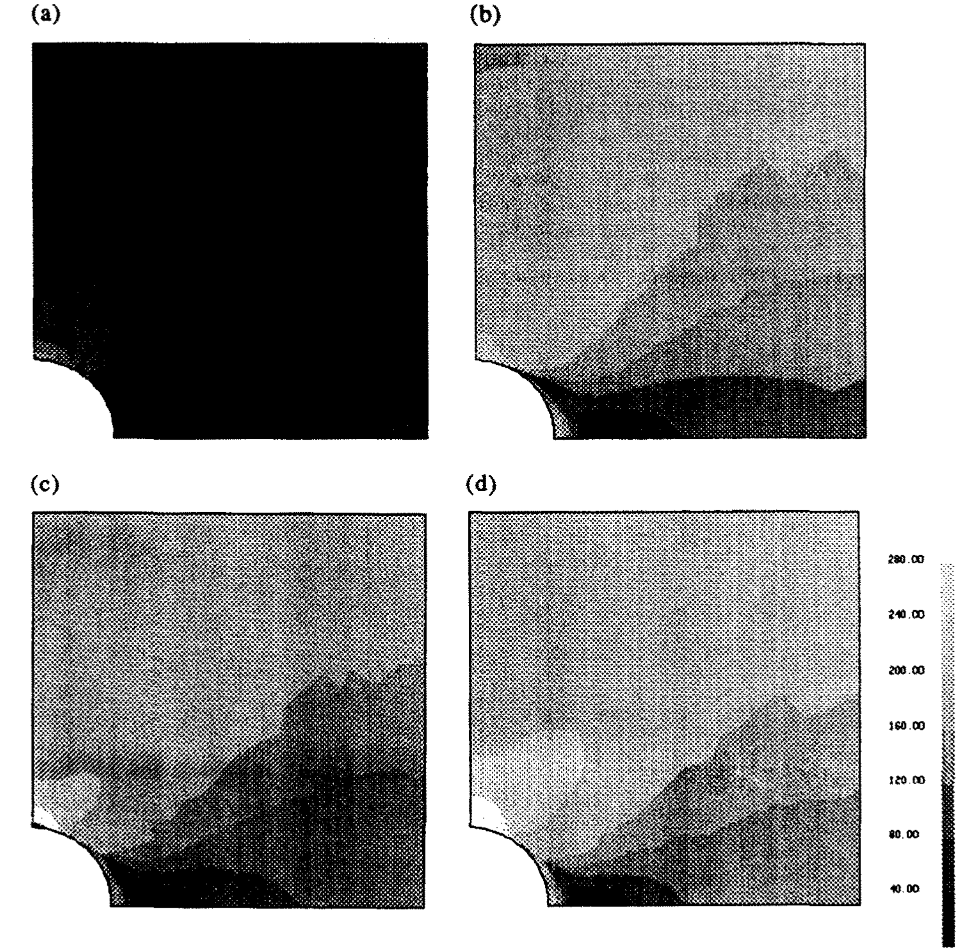

Fig. 7. Maps of the von Mises equivalent stress given by the domain decomposition method for an applied displacement of 0.0036 (a), 0.0358 (b), 0.0679 (c), and 0.01 (d).

(iv) The proposed algorithm makes good use of the parallel structure of computers. The subdomains have been numbered to avoid the need to simultaneously analyse overlapping subdomains during the iterations. The tests have shown that more than 90% of the computations can be parallelized with this approach.

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